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2000 J. Phys. A: Math. Gen. 33 1351

(http://iopscience.iop.org/0305-4470/33/7/305)

J. Phys. A: Math. Gen. **33** (2000) 1351–1361. Printed in the UK PII: S0305-4470(00)09258-1

Operator-valued Riemann–Hilbert problem for correlation functions of the XXZ spin chain

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Received 3 November 1999

Abstract. The generating functional of correlation functions for the XXZ spin chain is considered in the thermodynamic limit. We derive a system of integro-difference equations that prescribe this functional. On the basis of this system we establish the operator-valued Riemann–Hilbert problem for correlation functions of the XXZ spin chain.

1. Introduction

It has been shown by Korepin *et al* that many important problems arising from an analysis of correlation functions of quantum solvable models are reduced to classical inverse scattering problems [1]. In particular, for the XXX spin chain, the generating functional of correlation functions is represented by the Fredholm determinants [2], and a special correlation function, the so-called *ferromagnetic string formation probability* (FSFP), is proved to be connected with an operator-valued Riemann–Hilbert problem [3]. With the help of the solutions of this problem one can compute the large-distance asymptotic form of the FSFP [4]. Physically, this makes clear the probability of finding a ferromagnetic string of adjacent parallel spins for a given value of the magnetic field in the XXX spin chain.

In this paper, we consider the spin-$\frac{1}{2}$ XXZ Heisenberg chain in the thermodynamic limit. The Hamiltonian is defined by

$$H_{XXZ} = \sum_{n\in\mathbb{Z}} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cos 2\eta \sigma_n^z \sigma_{n+1}^z - h\sigma_n^z) \tag{1.1}$$

where σ_n^x, σ_n^y and σ_n^z are the Pauli matrices acting on the nth site and h is an external magnetic field. The anisotropy $\cos 2\eta$ implies the critical regime. Any correlation function of the model can be obtained by means of the generating functional $Q^{(m)}(\alpha)$ [5]. For example, a two-point correlation function $\langle \sigma_m^z \sigma_1^z \rangle$ is given by

$$\langle \sigma_m^z \sigma_1^z \rangle = 2\Delta_m \left. \frac{\partial^2 Q^{(m)}(\alpha)}{\partial^2 \alpha} \right|_{\alpha=0} - 4 \left. \frac{\partial Q^{(1)}(\alpha)}{\partial \alpha} \right|_{\alpha=0} - 1 \tag{1.2}$$

with the lattice Laplacian defined by $\Delta_m f(m) = f(m) - 2f(m-1) + f(m-2)$. This generating functional is represented by the Fredholm determinants,

$$Q^{(m)}(\alpha) = \frac{\langle \mathrm{vac} | \det \left(1 - V^{(m)}/2\pi\right) | \mathrm{vac} \rangle}{\langle \mathrm{vac} | \det \left(1 - (V^{(m)}/2\pi)|_{\alpha=0}\right) | \mathrm{vac} \rangle}. \tag{1.3}$$

Here the kernel is of the form

$$V^{(m)}(\lambda,\mu)=\frac{\sin 2\eta}{\sinh(\lambda-\mu)}\left(\frac{\exp(\alpha-\varphi_3(\lambda)+\varphi_4(\mu))e_1^{-1}(\lambda)e_1(\mu)+1}{\sinh(\lambda-\mu+2\mathrm{i}\eta)}\right.$$
$$\left.-\frac{e_2^{-1}(\lambda)e_2(\mu)+\exp(\alpha-\varphi_3(\lambda)+\varphi_4(\mu))}{\sinh(\mu-\lambda+2\mathrm{i}\eta)}\right). \tag{1.4}$$

From now on the index m is omitted if unnecessary. The integration contour that appears in the Fredholm determinants lies on the real axis $[-\Lambda,\Lambda]$, where a boundary value Λ is called the Fermi energy and depends on the anisotropy parameter η and the magnetic field h. The operators $e_1(\lambda)$ and $e_2(\lambda)$ are defined by

$$e_1(\lambda)=\left(\frac{\sinh(\lambda+\mathrm{i}\eta)}{\sinh(\lambda-\mathrm{i}\eta)}\right)^m \mathrm{e}^{-\varphi_1(\lambda)} \qquad e_2(\lambda)=\left(\frac{\sinh(\lambda+\mathrm{i}\eta)}{\sinh(\lambda-\mathrm{i}\eta)}\right)^m \mathrm{e}^{\varphi_2(\lambda)}. \tag{1.5}$$

The operators $\varphi_j(\lambda)$ $(j=1,\ldots,4)$ are bosonic quantum fields called the *dual fields*. They are decomposed into the momentum and coordinate fields,

$$\varphi_j(\lambda)=p_j(\lambda)+q_j(\lambda) \qquad (j=1,\ldots,4) \tag{1.6}$$

whose commutation relations are

$$\begin{aligned}&[p_j(\lambda),p_k(\mu)]=[q_j(\lambda),q_k(\mu)]=0\\ &[p_j(\lambda),q_k(\mu)]=U_{jk}h(\lambda,\mu)+U_{kj}h(\mu,\lambda) \qquad (j,k=1,\ldots,4)\end{aligned} \tag{1.7}$$

with

$$U=-\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&1&1&1\\1&0&1&1\end{pmatrix} \qquad h(\lambda,\mu)=\log\frac{\sinh(\lambda-\mu+2\mathrm{i}\eta)}{\mathrm{i}\sin 2\eta}. \tag{1.8}$$

Note that the dual fields are commutative: $[\varphi_j(\lambda),\varphi_k(\mu)]=0$ $(j,k=1,\ldots,4)$. They only produce vacuum expectation values according to the vacuum states defined by

$$\langle\mathrm{vac}|\mathrm{vac}\rangle=1 \qquad \langle\mathrm{vac}|q_j(\lambda)=p_j(\lambda)|\mathrm{vac}\rangle=0 \qquad (j=1,\ldots,4). \tag{1.9}$$

A special correlation function FSFP for the XXZ spin chain is defined by $P(m)=Q^{(m)}(-\infty)$, where m gives the length of a ferromagnetic string [5, 6]. In this case the expression of the kernel (1.4) is reduced to a simple form and enables us to formulate the related operator-valued Riemann–Hilbert problem in the same way as for the XXX spin chain [6]. Furthermore, in the limit of strong magnetic field $h\to h_c=4\cos^2\eta$, the kernel is related to the τ-function of the Painlevé V equation [7]. On the basis of this fact we can evaluate any correlation function of the XXZ spin chain under an external strong magnetic field. Similar results hold in the asymmetric XXZ spin chain that is a non-Hermitian generalization of the XXZ spin chain [8, 9].

The aim of the paper is to derive a system of integro-difference equations that prescribe the generating functional $Q^{(m)}(\alpha)$ and to establish the associated operator-valued Riemann–Hilbert problem. We *do not* take either the FSFP limit ($\alpha\to-\infty$) or the strong magnetic field limit ($h\to h_c$), and treat directly the kernel (1.4) that contains the dual fields with four species. This kernel is not of the desired type; however, it can be transformed into an integral operator that satisfies the integrable condition (see (2.7)). Thus the computation of the generating functional can be reduced to the solution of operator-valued Riemann–Hilbert problems. In section 2 we rewrite the kernel (1.4) and introduce some integral operators. In particular, the resolvent is expressed by products of two vectors (see lemma 1). In section 3 a system of integro-difference equations is derived. On the basis of this system we establish the operator-valued Riemann–Hilbert problem associated with the generating functional (1.3) in section 4. Section 5 is devoted to concluding remarks.

2. Integral operators

In this section we transform the kernel (1.4) into an integral operator that satisfies the integrable condition (2.7) and introduce some useful integral operators.

Definition 1. *We rewrite the kernel for the Fredholm determinant representation of the generating functional (1.3) as*

$$V(x,y) = \mathrm{i}\int_0^\infty \frac{\mathrm{d}s}{x-y}\sum_{j=1}^4 a_j(x|s)b_j(y|s) \tag{2.1}$$

where vectors $\boldsymbol{a}(x|s)$ and $\boldsymbol{b}(x|s)$ are defined by

$$\boldsymbol{a}(x|s) = \left(\mathrm{i}(q-q^{-1})\frac{qx-1}{x-q}\right)^{1/2}\begin{pmatrix} x^{m/2}\exp\left(-\mathrm{i}q\frac{qx-1}{x-q}s\right) \\ x^{-m/2}\exp\left(-\mathrm{i}q\frac{qx-1}{x-q}s+\alpha+\varphi_1(x)-\varphi_3(x)\right) \\ x^{-m/2}\exp\left(\mathrm{i}q^{-1}\frac{qx-1}{x-q}s-\varphi_2(x)\right) \\ x^{m/2}\exp\left(\mathrm{i}q^{-1}\frac{qx-1}{x-q}s+\alpha-\varphi_3(x)\right)\end{pmatrix} \tag{2.2}$$

$$\boldsymbol{b}^t(x|s) = \left(\mathrm{i}(q-q^{-1})\frac{qx-1}{x-q}\right)^{1/2}\begin{pmatrix} -x^{-m/2}\exp\left(\mathrm{i}q^{-1}\frac{qx-1}{x-q}s\right) \\ -x^{m/2}\exp\left(\mathrm{i}q^{-1}\frac{qx-1}{x-q}s-\varphi_1(x)+\varphi_4(x)\right) \\ x^{m/2}\exp\left(-\mathrm{i}q\frac{qx-1}{x-q}s+\varphi_2(x)\right) \\ x^{-m/2}\exp\left(-\mathrm{i}q\frac{qx-1}{x-q}s+\varphi_4(x)\right)\end{pmatrix}. \tag{2.3}$$

The superscript t indicates the transposition of a vector. The integration contour for variables x, y runs anticlockwise on the unit circle: $C : x = \mathrm{e}^{\mathrm{i}\theta}$ $(-\psi < \theta < 2\pi + \psi)$. The endpoints are denoted by $\zeta_\pm = \mathrm{e}^{\pm\mathrm{i}\psi}$. The dual fields still obey the commutation relations (1.7) with

$$h(x,y) = \log\left[\frac{1}{q-q^{-1}}\left(q\sqrt{\frac{qx-1}{x-q}\frac{y-q}{qy-1}} - q^{-1}\sqrt{\frac{x-q}{qx-1}\frac{qy-1}{y-q}}\right)\right]. \tag{2.4}$$

The original kernel (1.4) is obtained by using the transformations

$$x = \frac{\sinh(\lambda+\mathrm{i}\eta)}{\sinh(\lambda-\mathrm{i}\eta)} \qquad q = \mathrm{e}^{2\mathrm{i}\eta} \qquad \psi = \mathrm{i}\log\frac{\sinh(\Lambda-\mathrm{i}\eta)}{\sinh(\Lambda+\mathrm{i}\eta)} \tag{2.5}$$

and an identity that is valid for $\frac{1}{2}\pi < \eta < \pi$:

$$\int_0^\infty \mathrm{d}s \exp\left(-\mathrm{i}q\frac{qx-1}{x-q}s + \mathrm{i}q^{-1}\frac{qy-1}{y-q}s\right) = -\mathrm{i}\left(q\frac{qx-1}{x-q} - q^{-1}\frac{qy-1}{y-q}\right)^{-1}. \tag{2.6}$$

Here an extra factor that leaves the Fredholm determinants unchanged appears; however, it can be ignored because it has no influence on the representation of the generating functional (1.3). We remark that the vectors $\boldsymbol{a}(x|s)$ and $\boldsymbol{b}(x|s)$ satisfy

$$\int_0^\infty \mathrm{d}s \sum_{j=1}^4 a_j(x|s) b_j(x|s) = 0. \tag{2.7}$$

This relation is called the *integrable condition* and ensures that the kernel (2.1) is well defined for any $x, y \in \mathbb{C}$. By virtue of this condition we can formulate the operator-valued Riemann–Hilbert problem in section 4.

Hereafter some matrices that contain integral operators appear. We call them the *matrix integral operators* and denote them by bold letters, except that '1' indicates the product of the delta-function and the unit matrix. The following convention for matrix integral operators is adopted unless it causes confusion:

$$(\boldsymbol{EF})_{jk}(x, y|s, t) = \int_C \mathrm{d}z \int_0^\infty \mathrm{d}r \sum_n E_{jn}(x, z|s, r) F_{nk}(z, y|r, t). \tag{2.8}$$

Definition 2. *We introduce the vectors* $\boldsymbol{\alpha}(x|s)$ *and* $\boldsymbol{\beta}(x|s)$ *via*

$$\left(\left(1 - \frac{1}{2\pi}\boldsymbol{V}\right)\boldsymbol{\alpha}\right)(x|s) = \boldsymbol{a}(x|s) \qquad \left(\boldsymbol{\beta}\left(1 - \frac{1}{2\pi}\boldsymbol{V}\right)\right)(x|s) = \boldsymbol{b}(x|s) \tag{2.9}$$

and define the resolvent $R(x, y)$ *through*

$$\left(1 - \frac{1}{2\pi}V\right)\left(1 + \frac{1}{2\pi}R\right) = \left(1 + \frac{1}{2\pi}R\right)\left(1 - \frac{1}{2\pi}V\right) = 1. \tag{2.10}$$

Note that one of these relations can be derived by letting $(1 + R/2\pi)$ act on both sides of the other relation from the left or the right. Thus $(1 - V/2\pi)$ and $(1 + R/2\pi)$ are commutative.

Lemma 1. *The resolvent is represented by*

$$R(x, y) = \mathrm{i}\int_0^\infty \frac{\mathrm{d}s}{x-y} \sum_{j=1}^4 \alpha_j(x|s)\beta_j(y|s). \tag{2.11}$$

Proof. The defining relation of the resolvent is rewritten as $(1 - V/2\pi)R = V$. By using the identity $(x-y) = (x-z) + (z-y)$ this relation is deformed into

$$(x-y)R(x,y) - \frac{\mathrm{i}}{2\pi}\int_C \mathrm{d}z \int_0^\infty \mathrm{d}s \sum_{j=1}^4 a_j(x|s) b_j(z|s) R(z,y)$$
$$- \frac{1}{2\pi}\int_C \mathrm{d}z\, V(x,z)(z-y)R(z,y) = \mathrm{i}\int_0^\infty \mathrm{d}s \sum_{j=1}^4 a_j(x|s) b_j(y|s). \tag{2.12}$$

Move the second term in the left-hand side to the right-hand side and recall the defining relations (2.9) and (2.10). The representation (2.11) is thus obtained. □

Definition 3. *We define the* 4×4 *matrix integral operators with variables* s, t *by*

$$A_{jk}(s,t) = \mathrm{i}\int_C \frac{\mathrm{d}x}{x}\,\alpha_j(x|s)b_k(x|t) \tag{2.13}$$

$$B_{jk}(s,t) = \mathrm{i}\int_C \frac{\mathrm{d}x}{x}\,a_j(x|s)\beta_k(x|t) \qquad (j,k = 1,\ldots,4). \tag{2.14}$$

Lemma 2. *The integral operator* $\boldsymbol{B}(s,t)$ *is the resolvent of* $\boldsymbol{A}(s,t)$*. Namely,*

$$\left(1 - \frac{1}{2\pi}\boldsymbol{A}\right)\left(1 + \frac{1}{2\pi}\boldsymbol{B}\right) = \left(1 + \frac{1}{2\pi}\boldsymbol{B}\right)\left(1 - \frac{1}{2\pi}\boldsymbol{A}\right) = 1. \tag{2.15}$$

Proof. The action of $(1 - V/2\pi)$ on $\boldsymbol{\alpha}(x|s)/x$ is computed as

$$\begin{aligned}\frac{1}{x}\boldsymbol{\alpha}(x|s) - \frac{1}{2\pi}\int_C \frac{\mathrm{d}y}{y}\,V(x,y)\boldsymbol{\alpha}(y|s) &= \frac{1}{x}\boldsymbol{a}(x|s) + \frac{1}{2\pi}\int_C \mathrm{d}y\left(\frac{1}{x} - \frac{1}{y}\right)V(x,y)\boldsymbol{\alpha}(y|s)\\ &= \frac{1}{x}\left(\left(1 - \frac{1}{2\pi}\boldsymbol{A}\right)\boldsymbol{a}\right)(x|s).\end{aligned} \tag{2.16}$$

By virtue of this relation we obtain

$$\begin{aligned}A_{jk}(s,t) &= \mathrm{i}\int_C \frac{\mathrm{d}x}{x}\,\alpha_j(x|s)\left(\beta_k\left(1 - \frac{1}{2\pi}V\right)\right)(x|t)\\ &= \mathrm{i}\int_C \mathrm{d}x\,\beta_k(x|t)\left(\frac{1}{x}\alpha_j(x|s) - \frac{1}{2\pi}\int_C \frac{\mathrm{d}y}{y}V(x,y)\alpha_j(y|s)\right)\\ &= \mathrm{i}\int_C \frac{\mathrm{d}x}{x}\,\beta_k(x|t)\left(\left(1 - \frac{1}{2\pi}\boldsymbol{A}\right)\boldsymbol{a}\right)_j(x|s)\\ &= \left(\left(1 - \frac{1}{2\pi}\boldsymbol{A}\right)\boldsymbol{B}\right)_{jk}(s,t) \qquad (j,k = 1,\ldots,4).\end{aligned} \tag{2.17}$$

This means that $(1 - \boldsymbol{A}/2\pi)\boldsymbol{B} = \boldsymbol{A}$ and therefore implies the lemma. The proof has been completed. □

Applying lemma 2 to (2.16) we can obtain another representation of $\boldsymbol{a}(x|s)/x$:

$$\frac{1}{x}\boldsymbol{a}(x|s) = \frac{1}{x}\left(\left(1 + \frac{1}{2\pi}\boldsymbol{B}\right)\boldsymbol{\alpha}\right)(x|s) - \frac{1}{2\pi}\int_C \frac{\mathrm{d}y}{y}V(x,y)\left(\left(1 + \frac{1}{2\pi}\boldsymbol{B}\right)\boldsymbol{\alpha}\right)(y|s). \tag{2.18}$$

We remark that the kernel $V(x,y)$ commutes with $\boldsymbol{A}(s,t)$ and $\boldsymbol{B}(s,t)$ because the variables are different. This relation plays an important role in the next section.

3. Integro-difference equations

In order to make clear the properties of the unknown vectors $\boldsymbol{\alpha}^{(m)}(x|s)$ and $\boldsymbol{\beta}^{(m)}(x|s)$, we compute their dependences on m and ψ, and derive a system of integro-difference equations for them. This helps us to formulate an operator-valued Riemann–Hilbert problem for the XXZ spin chain.

Lemma 3. *The vectors* $\boldsymbol{\alpha}^{(m)}(x|s)$ *and* $\boldsymbol{\beta}^{(m)}(x|s)$ *obey the following integro-difference equations with respect to* m*:*

$$\boldsymbol{\alpha}^{(m+1)}(x|s) = \left(\left(\sqrt{x}\gamma_1 + \frac{1}{\sqrt{x}}\left(1 - \frac{1}{2\pi}\boldsymbol{A}^{(m+1)}\right)\gamma_2\left(1 + \frac{1}{2\pi}\boldsymbol{B}^{(m)}\right)\right)\boldsymbol{\alpha}^{(m)}\right)(x|s) \tag{3.1}$$

$$\boldsymbol{\beta}^{(m+1)}(x|s) = \left(\boldsymbol{\beta}^{(m)}\left(\sqrt{x}\gamma_2 + \frac{1}{\sqrt{x}}\left(1 - \frac{1}{2\pi}\boldsymbol{A}^{(m)}\right)\gamma_1\left(1 + \frac{1}{2\pi}\boldsymbol{B}^{(m+1)}\right)\right)\right)(x|s) \tag{3.2}$$

where γ_1 *and* γ_2 *are* 4×4 *diagonal matrices*

$$\gamma_1 = \tfrac{1}{2}(1 + \sigma^z \otimes \sigma^z) \qquad \gamma_2 = \tfrac{1}{2}(1 - \sigma^z \otimes \sigma^z). \tag{3.3}$$

Proof. By the definitions of $\boldsymbol{a}^{(m)}(x|s)$ and $\boldsymbol{b}^{(m)}(x|s)$ their dependences on m are

$$\boldsymbol{a}^{(m+1)}(x|s) = \left(\sqrt{x}\gamma_1 + \frac{1}{\sqrt{x}}\gamma_2\right)\boldsymbol{a}^{(m)}(x|s) \tag{3.4}$$

$$\boldsymbol{b}^{(m+1)}(x|s) = \boldsymbol{b}^{(m)}(x|s)\left(\sqrt{x}\gamma_2 + \frac{1}{\sqrt{x}}\gamma_1\right). \tag{3.5}$$

Similarly the kernel $V^{(m+1)}(x, y)$ is connected with $V^{(m)}(x, y)$ as

$$V^{(m+1)}(x, y) = \sqrt{\frac{x}{y}}\left(V^{(m)}(x, y) - \mathrm{i}\int_0^\infty \frac{\mathrm{d}s}{x}\,\boldsymbol{b}^{(m)}(y|s)\gamma_2\boldsymbol{a}^{(m)}(x|s)\right). \tag{3.6}$$

Using these recursion relations we obtain

$$\begin{aligned}\boldsymbol{\alpha}^{(m+1)}(x|s) - \boldsymbol{a}^{(m+1)}(x|s) &= \frac{1}{2\pi}(V^{(m+1)}\boldsymbol{\alpha}^{(m+1)})(x|s)\\ &= \frac{\sqrt{x}}{2\pi}\int_C \frac{\mathrm{d}y}{\sqrt{y}}\,V^{(m)}(x, y)\boldsymbol{\alpha}^{(m+1)}(y|s)\\ &\quad -\frac{1}{2\pi\sqrt{x}}(\boldsymbol{A}^{(m+1)}\gamma_2\boldsymbol{a}^{(m)})(x|s).\end{aligned} \tag{3.7}$$

It thus follows that

$$\begin{aligned}&\frac{1}{\sqrt{x}}\boldsymbol{\alpha}^{(m+1)}(x|s) - \frac{1}{2\pi}\int_C \frac{\mathrm{d}y}{\sqrt{y}}V^{(m)}(x, y)\boldsymbol{\alpha}^{(m+1)}(y|s)\\ &\quad = \frac{1}{\sqrt{x}}\boldsymbol{a}^{(m+1)}(x|s) - \frac{1}{2\pi x}(\boldsymbol{A}^{(m+1)}\gamma_2\boldsymbol{a}^{(m)})(x|s)\\ &\quad = \gamma_1\boldsymbol{a}^{(m)}(x|s) + \frac{1}{x}\left(\left(1 - \frac{1}{2\pi}\boldsymbol{A}^{(m+1)}\right)\gamma_2\boldsymbol{a}^{(m)}\right)(x|s)\\ &\quad = \left(\left(1 - \frac{1}{2\pi}V^{(m)}\right)\gamma_1\boldsymbol{\alpha}^{(m)}\right)(x|s)\\ &\qquad + \frac{1}{x}\left(\left(1 - \frac{1}{2\pi}\boldsymbol{A}^{(m+1)}\right)\gamma_2\left(1 + \frac{1}{2\pi}\boldsymbol{B}^{(m)}\right)\boldsymbol{\alpha}^{(m)}\right)(x|s)\\ &\qquad - \frac{1}{2\pi}\int_C \frac{\mathrm{d}y}{y}V^{(m)}(x, y)\left(\left(1 - \frac{1}{2\pi}\boldsymbol{A}^{(m+1)}\right)\gamma_2\left(1 + \frac{1}{2\pi}\boldsymbol{B}^{(m)}\right)\boldsymbol{\alpha}^{(m)}\right)(y|s).\end{aligned} \tag{3.8}$$

In the last equality the relation (2.18) is used. Thus the integro-difference equation (3.1) is obtained by removing the action of $(1 - V^{(m)}/2\pi)$. In the same way (3.2) can be derived. □

Lemma 4. *The derivatives of the vectors* $\boldsymbol{\alpha}(x|s)$ *and* $\boldsymbol{\beta}(x|s)$ *with respect to* ψ *are*

$$\partial_\psi\boldsymbol{\alpha}(x|s) = -\frac{1}{2\pi}\sum_{\varepsilon=\pm}\left(\frac{\zeta_\varepsilon}{x - \zeta_\varepsilon}\boldsymbol{\alpha}(\zeta_\varepsilon|s)\int_0^\infty \mathrm{d}t\sum_{j=1}^4 \alpha_j(x|t)\beta_j(\zeta_\varepsilon|t)\right) \tag{3.9}$$

$$\partial_\psi\boldsymbol{\beta}(x|s) = -\frac{1}{2\pi}\sum_{\varepsilon=\pm}\left(\frac{\zeta_\varepsilon}{\zeta_\varepsilon - x}\boldsymbol{\beta}(\zeta_\varepsilon|s)\int_0^\infty \mathrm{d}t\sum_{j=1}^4 \alpha_j(\zeta_\varepsilon|t)\beta_j(x|t)\right). \tag{3.10}$$

Proof. We show the differential equation (3.9). The other equation (3.10) can be derived in the same way. According to the identity

$$-\mathrm{i}\partial_\psi \int_C \mathrm{d}x\, f(x) = \sum_{\varepsilon=\pm} \zeta_\varepsilon f(\zeta_\varepsilon) \tag{3.11}$$

the differentiation of (2.9) is computed as

$$\partial_\psi \boldsymbol{\alpha}(x|s) - \frac{\mathrm{i}}{2\pi} \sum_{\varepsilon=\pm} \zeta_\varepsilon V(x, \zeta_\varepsilon)\boldsymbol{\alpha}(\zeta_\varepsilon|s) - \frac{1}{2\pi}(V\partial_\psi \boldsymbol{\alpha})(x|s) = 0. \tag{3.12}$$

By using the resolvent this is reduced to

$$\partial_\psi \boldsymbol{\alpha}(x|s) = \frac{\mathrm{i}}{2\pi} \sum_{\varepsilon=\pm} \zeta_\varepsilon R(x, \zeta_\varepsilon)\boldsymbol{\alpha}(\zeta_\varepsilon|s). \tag{3.13}$$

Because of lemma 1 we obtain (3.9). □

The following lemma is not directly related to the formulation of operator-valued Riemann–Hilbert problems but is useful for evaluating the large-m asymptotic form of the generating functional $Q^{(m)}(\alpha)$.

Lemma 5. *The Fredholm determinant* $\det(1 - V^{(m)}/2\pi)$ *satisfies the following recursion relation with respect to* m*:*

$$\frac{\det\left(1 - V^{(m+1)}/2\pi\right)}{\det\left(1 - V^{(m)}/2\pi\right)} = \exp \operatorname{tr} \log\left(1 + \frac{1}{2\pi}\gamma_2 \boldsymbol{B}^{(m)}\right) \tag{3.14}$$

where the trace for matrix integral operators is defined by

$$\operatorname{tr}(\boldsymbol{K}^n) = \int_0^\infty \prod_{k=1}^n \mathrm{d}s_k \sum_{j_1,\ldots,j_n=1}^4 K_{j_1,j_2}(s_1, s_2) \cdots K_{j_n,j_1}(s_n, s_1). \tag{3.15}$$

Proof. Due to the relation (3.6) it follows that

$$\begin{aligned}
\delta(x-y) &- \frac{1}{2\pi}\sqrt{\frac{y}{x}} V^{(m+1)}(x, y) \\
&= \delta(x-y) - \frac{1}{2\pi}\left(V^{(m)}(x, y) - \mathrm{i}\int_0^\infty \frac{\mathrm{d}s}{x}\left(\boldsymbol{\beta}^{(m)}\left(1 - \frac{1}{2\pi}V^{(m)}\right)\right)(y|s)\gamma_2 \boldsymbol{a}^{(m)}(x|s)\right) \\
&= \left(\left(1 + \frac{1}{2\pi}G^{(m)}\right)\left(1 - \frac{1}{2\pi}V^{(m)}\right)\right)(x, y)
\end{aligned} \tag{3.16}$$

where

$$G^{(m)}(x, y) = \mathrm{i}\int_0^\infty \frac{\mathrm{d}s}{x}\, \boldsymbol{\beta}^{(m)}(y|s)\gamma_2 \boldsymbol{a}^{(m)}(x|s). \tag{3.17}$$

Based on this recursion relation the Fredholm determinants of both sides are given by

$$\det\left(1 - \frac{1}{2\pi}V^{(m+1)}\right) = \det\left(1 + \frac{1}{2\pi}G^{(m)}\right)\det\left(1 - \frac{1}{2\pi}V^{(m)}\right). \tag{3.18}$$

Here we note that $\sqrt{y/x}$ in the left-hand side has no influence on the Fredholm determinant. Take the logarithm of $\det(1 + G/2\pi)$:

$$\log\det\left(1 + \frac{1}{2\pi}G\right) = \operatorname{tr}\log\left(1 + \frac{1}{2\pi}G\right) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n(2\pi)^n}\operatorname{tr}(G^n). \tag{3.19}$$

The trace of G^n can be expressed in terms of $\boldsymbol{B}$:

$$
\begin{aligned}
\operatorname{tr}(G^n) &= \mathrm{i}^n \int_C \prod_{k=1}^n \frac{\mathrm{d}x_k}{x_k} \int_0^\infty \prod_{k=1}^n \mathrm{d}s_k \\
&\quad \times \sum_{j_1,\ldots,j_n=2,3} a_{j_1}(x_1|s_1)\beta_{j_1}(x_2|s_1)a_{j_2}(x_2|s_2)\beta_{j_2}(x_3|s_2)\cdots a_{j_n}(x_n|s_n)\beta_{j_n}(x_1|s_n) \\
&= \int_0^\infty \prod_{k=1}^n \mathrm{d}s_k \sum_{j_1,\ldots,j_n=2,3} B_{j_1,j_n}(s_1,s_n)B_{j_2,j_1}(s_2,s_1)\cdots B_{j_n,j_{n-1}}(s_n,s_{n-1}) \\
&= \operatorname{tr}((\gamma_2 \boldsymbol{B})^n).
\end{aligned} \tag{3.20}
$$

Equation (3.18) with (3.19) and (3.20) proves the lemma. □

4. Operator-valued Riemann–Hilbert problem

In this section we establish the operator-valued Riemann–Hilbert problem associated with the generating functional (1.3). Let us consider the 4×4 matrix integral operator $\boldsymbol{\chi}^{(m)}(z|s,t)$ that has the following properties. Hereafter not only the index m but also variables s, t are omitted if unnecessary.

(a) $\boldsymbol{\chi}(z)$ is analytic for $z \in \mathbb{C}\backslash C$.

(b) Let $\boldsymbol{\chi}_{\text{ext}}(z)$ and $\boldsymbol{\chi}_{\text{int}}(z)$ be the limits of $\boldsymbol{\chi}(z)$ from outside and inside of the unit circle, respectively. It then follows that

$$\boldsymbol{\chi}_{\text{ext}}(z) = \boldsymbol{\chi}_{\text{int}}(z)\boldsymbol{L}(z) \qquad (z \in C) \tag{4.1}$$

where the *conjugation matrix* $\boldsymbol{L}(z)$ is given by

$$\boldsymbol{L}(z) = 1 - \boldsymbol{l}(z) \qquad l_{jk}(z|s,t) = a_j(z|s)b_k(z|t) \qquad (j,k = 1,\ldots,4). \tag{4.2}$$

(c) $\boldsymbol{\chi}(\infty) = 1$.

We point out that the conjugation matrix $\boldsymbol{L}(z)$ is the 4×4 matrix integral operator with variables s, t. For example, the condition (4.1) means

$$(\chi_{\text{ext}})_{jk}(z|s,t) = \int_0^\infty \mathrm{d}r \sum_{n=1}^4 (\chi_{\text{int}})_{jn}(z|s,r)L_{nk}(z|r,t) \qquad (j,k = 1,\ldots,4). \tag{4.3}$$

The connection of the operator-valued Riemann–Hilbert problem (a)–(c) to lemmas 1–5 in the previous section is summarized in the following theorem.

Theorem. *Suppose that the solution of the Riemann–Hilbert problem (a)–(c) exists and is unique. Define new vectors by*

$$\boldsymbol{\alpha}(z|s) = \int_0^\infty \mathrm{d}t\, \boldsymbol{\chi}(z|s,t)\boldsymbol{a}(z|t) \qquad \boldsymbol{\beta}(z|s) = \int_0^\infty \mathrm{d}t\, \boldsymbol{b}(z|t)\boldsymbol{\chi}^{-1}(z|t,s) \tag{4.4}$$

and introduce the integral operators $\boldsymbol{A}(s,t)$ and $\boldsymbol{B}(s,t)$ as definition 3 again. Then they satisfy lemmas 2–4.

Proof. We start from the proof of lemma 2. Using the definition of $\boldsymbol{A}(s,t)$ and the *canonical integral representation* of $\boldsymbol{\chi}(z)$ (see [1, 3, 6])

$$\boldsymbol{\chi}(z) = 1 + \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{d}\zeta}{\zeta - z} \boldsymbol{\chi}_{\text{int}}(\zeta)\boldsymbol{l}(\zeta) \tag{4.5}$$

we obtain

$$\chi(0) = 1 - \frac{\mathrm{i}}{2\pi}\int_C \frac{\mathrm{d}z}{z}\,\chi_{\mathrm{int}}(z)\boldsymbol{l}(z) = 1 - \frac{1}{2\pi}\boldsymbol{A}. \tag{4.6}$$

Since $\boldsymbol{l}^2(z) = 0$ (which is clear from the integrable condition (2.7)) there exists an inverse of the conjugation matrix: $\boldsymbol{L}^{-1}(z) = 1 + \boldsymbol{l}(z)$. The canonical integral representation of $\chi^{-1}(z)$ is thus given by

$$\chi^{-1}(0) = 1 + \frac{\mathrm{i}}{2\pi}\int_C \frac{\mathrm{d}z}{z}\,\boldsymbol{l}(z)\chi_{\mathrm{int}}^{-1}(z) = 1 + \frac{1}{2\pi}\boldsymbol{B}. \tag{4.7}$$

The identity $\chi(0)\chi^{-1}(0) = \chi^{-1}(0)\chi(0) = 1$ corresponds to lemma 2.

Let us compute the integro-difference relation of $\chi^{(m)}(z)$ with respect to m and prove lemma 3. We introduce the following integral operator:

$$\boldsymbol{\Psi}^{(m)}(z) = \chi^{(m)}(z)(\gamma_1 + z^{-m}\gamma_2). \tag{4.8}$$

This obeys

$$\boldsymbol{\Psi}^{(m)}_{\mathrm{ext}}(z) = \boldsymbol{\Psi}^{(m)}_{\mathrm{int}}(z)\boldsymbol{L}_0(z) \qquad (z \in C) \tag{4.9}$$

with the conjugation matrix

$$\boldsymbol{L}_0(z) = (\gamma_1 + z^m\gamma_2)\boldsymbol{L}^{(m)}(z)(\gamma_1 + z^{-m}\gamma_2). \tag{4.10}$$

From the definitions of $\boldsymbol{a}^{(m)}(z)$, $\boldsymbol{b}^{(m)}(z)$ and $\boldsymbol{L}^{(m)}(z)$, we see that $\boldsymbol{L}_0(z)$ is independent of m. Hence applying Liouville's theorem we have

$$\boldsymbol{\Psi}^{(m+1)}(z)(\boldsymbol{\Psi}^{(m)}(z))^{-1} = \gamma_1 + \frac{1}{z}\chi^{(m+1)}(0)\gamma_2(\chi^{(m)}(0))^{-1} \tag{4.11}$$

which implies

$$\chi^{(m+1)}(z)\left(\sqrt{z}\gamma_1 + \frac{1}{\sqrt{x}}\gamma_2\right) = \left(\sqrt{z}\gamma_1 + \frac{1}{\sqrt{x}}\chi^{(m+1)}(0)\gamma_2(\chi^{(m)}(0))^{-1}\right)\chi^{(m)}(z). \tag{4.12}$$

By letting $\boldsymbol{a}^{(m)}(z|s)$ act on this relation from the right the integro-difference equation (3.1) is derived. Similarly (3.2) is obtained. Thus lemma 3 has been proved.

We show lemma 4. In the same way as for the FSFP case [6] and the impenetrable Bose gas case [10], in the neighbourhood of C, the integral operator $\chi(z)$ can be decomposed into

$$\chi(z) = \widehat{\chi}(z)\chi_0(z) \tag{4.13}$$

where $\widehat{\chi}(z)$ is a single-valued, invertible and analytic in the neighbourhood of C. $\chi_0(z)$ is represented by

$$\chi_0(z) = 1 + \frac{\mathrm{i}}{2\pi}\log\frac{z-\zeta_-}{z-\zeta_+}\boldsymbol{l}(z). \tag{4.14}$$

Since $\boldsymbol{l}^2(z) = 0$ its logarithm-derivative is computed as

$$\partial_\psi\chi_0(z)\chi_0^{-1}(z) = -\frac{1}{2\pi}\sum_{\varepsilon=\pm}\frac{\zeta_\varepsilon}{z-\zeta_\varepsilon}\boldsymbol{l}(z). \tag{4.15}$$

Due to this relation and Liouville's theorem the logarithm-derivative of $\chi(z)$ is written as

$$\partial_\psi\chi(z)\chi^{-1}(z) = \sum_{\varepsilon=\pm}\frac{1}{z-\zeta_\varepsilon}\boldsymbol{X}_\varepsilon \tag{4.16}$$

with the coefficient $\boldsymbol{X}_{\pm}$,

$$\begin{aligned}\boldsymbol{X}_{\pm} &= \lim_{z\to\zeta_{\pm}} (z-\zeta_{\pm})\widehat{\boldsymbol{\chi}}(z)\partial_{\psi}\boldsymbol{\chi}_0(z)\boldsymbol{\chi}_0^{-1}(z)\widehat{\boldsymbol{\chi}}^{-1}(z) \\ &= -\frac{\zeta_{\pm}}{2\pi}\widehat{\boldsymbol{\chi}}(\zeta_{\pm})\boldsymbol{l}(\zeta_{\pm})\widehat{\boldsymbol{\chi}}^{-1}(\zeta_{\pm}) \\ &= -\frac{\zeta_{\pm}}{2\pi}\boldsymbol{\chi}(\zeta_{\pm})\boldsymbol{l}(\zeta_{\pm})\boldsymbol{\chi}^{-1}(\zeta_{\pm}).\end{aligned} \tag{4.17}$$

By definitions of $\boldsymbol{\alpha}(z|s)$ and $\boldsymbol{\beta}(z|s)$ the elements of $\boldsymbol{X}_{\pm}$ are expressed by

$$(\boldsymbol{X}_{\pm})_{jk}(s,t) = -\frac{\zeta_{\pm}}{2\pi}\alpha_j(\zeta_{\pm}|s)\beta_k(\zeta_{\pm}|t) \qquad (j,k=1,\ldots,4). \tag{4.18}$$

Let $\boldsymbol{\chi}(z)$ act on (4.16) from the right and use this representation of $\boldsymbol{X}_{\pm}$. The differential equation (3.9) is thus obtained. Similarly (3.10) can be derived. The proof of the theorem has been completed. □

As a consequence of the Riemann–Hilbert problem (a)–(c) lemmas 1 and 5 certainly hold. On the basis of these lemmas we can evaluate the large-m asymptotic behaviour of correlation functions for the XXZ spin chain.

5. Concluding remarks

In this paper we have derived a system of integro-difference equations that prescribe the generating functional of correlation functions and have established an operator-valued Riemann–Hilbert problem for the XXZ spin chain. It can be easily checked that, in the limit $\alpha \to -\infty$, our problem reduces to the problem associated with the FSFP obtained in [6]. We are in a position to evaluate the long-distance asymptotic behaviour of any correlation function for the XXZ spin chain. In a forthcoming publication our Riemann–Hilbert problem will be used to compute the large-m asymptotic form of the generating functional.

At the free-fermionic point $\eta = 3\pi/4$ the XXZ spin chain is reduced to the XXO model. The FSFP of the XXO model is known to be connected with a Riemann–Hilbert problem whose conjugation matrix is not an integral operator [11]. Its large-m asymptotic form is computed as follows:

$$P_{XXO}(m) \sim \left(\tfrac{1}{4}(h+2)\right)^{m^2/2} \tag{5.1}$$

where h is the magnetic field. It is interesting to reproduce the same result from our Riemann–Hilbert problem by taking the limit $\alpha \to -\infty$ and $\eta \to 3\pi/4$. Furthermore, we can evaluate any correlation function for the XXO model.

Operator-valued Riemann–Hilbert problems appear in several subjects, apart from the computation of correlation functions of quantum solvable models. For example, it is well known that the classical inverse problem for integrable equations in 2 + 1 dimensions (Davey–Stewartson, KP, etc) can be expressed as an operator-valued Riemann–Hilbert problem. The investigation of operator-valued Riemann–Hilbert problems is thus important and interesting from the standpoint of not only physics but also mathematics.

Acknowledgments

The authors wish to thank Professor V E Korepin and Dr T Tsuchida for useful comments.

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